Chapter 6 – Fourier Optics

Gabriel Popescu

University of Illinois at Urbana-Champaign
Beckman Institute

Quantitative Light Imaging Laboratory
http://light.ece.uiuc.edu
3.10 Lens as a phase transformer

- $E = E_o \cdot e^{i\phi}$ becomes important
- How is the wavefront changed by a lens?

\[
\phi(x, y) = knb(x, y) + k[b_o - b(x, y)]
\]

\[
= kb_o + k(n - 1)b(x, y)
\]
3.10 Lens as a phase transformer

- Let’s calculate $b(x,y)$; assume small angles
  
  \[
  b_1 = R_1 - (PC_1) = R_1 - \sqrt{R_1^2 - (\alpha R_1)^2} = R_1 \left[ 1 - \sqrt{1 - \alpha^2} \right]
  \]

- Taylor expansion: \( \sqrt{1 + x} \big|_{x \to \infty} \approx 1 + \frac{x}{2} \)
  
  \[\Rightarrow b_1 = R_1 \left[ 1 - \left( 1 - \frac{\alpha^2}{2} \right) \right] = R_1 \frac{\alpha^2}{2}\]

- \( \alpha = \tan \alpha = \frac{\sqrt{x^2 + y^2}}{R_1} \)

- So:
  
  \[
  b_1(x, y) = \frac{x^2 + y^2}{2R_1}
  \]
3.10 Lens as a phase transformer

$$b(x, y) = b_o - b_1(x, y) - b_2(x, y) =$$

$$= b_o - \frac{x^2 + y^2}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

(3.35)

- This is the thickness approximation
- The phase \( \phi \) becomes:

$$\phi(x, y) = \phi_o - k(n-1)b(x, y) =$$

$$= \phi_o - k \frac{x^2 + y^2}{2} (n-1) \left[ \frac{1}{R_1} - \frac{1}{R_2} \right]$$

(3.36)

- But we know:  
  $$\frac{1}{f} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$
3.10 Lens as a phase transformer

\[ E'(x, y) = E(x, y) \cdot t_e(x, y) \]  \hspace{1cm} (3.37)

- The lens transformation is:

\[
t_e = e^{i\phi} = e^{iknb} \cdot e^{-i\frac{k}{2f}(x^2+y^2)} \]  \hspace{1cm} (3.38)
3.10 Lens as a phase transformer

- A lens transforms an incident plane wavefront into a **parabolic shape**
- Note: \( f > 0 \) convergent lens
  \( f < 0 \) divergent

- So, if we know how to propagate through free space, then we can calculate field **amplitude** and **phase** through any imaging system
3.11 Huygens-Fresnel principle

- Spherical waves:
- Wavelet: \( h = \frac{e^{ikR}}{R} \)
- \( R = \sqrt{x^2 + y^2 + z^2} = z \sqrt{1 + \frac{x^2 + y^2}{z^2}} \)
- We are interested close to OA, i.e. small angles

\[
R \approx z \left[ 1 + \frac{1}{2} \left( \frac{x^2 + y^2}{z^2} \right) \right] \quad (3.39)
\]
3.11 Huygens-Fresnel principle

- For amplitude \( \frac{1}{R} \approx \frac{1}{z} \) is OK
- For phase \( kR \approx kz \left[ 1 + \frac{1}{z} \left( \frac{x^2 + y^2}{z^2} \right) \right] \)

→ The wavelet becomes:
\[
h(x, y) = \frac{e^{ikz}}{z} e^{i\frac{k(x^2+y^2)}{2z}}
\]  
(3.40a) \[ f(x, y) = x^2 + y^2 \]

Remember, for the lens we found:
\[
t_e(x, y) = e^{i\phi} e^{i\frac{k}{2f}(x^2+y^2)}
\]  
(3.40b) \[ f(x) = x^2 \]

- Free space acts on the wavefront like a divergent lens (note “+” sign in phase)
3.11 Huygens-Fresnel principle

- At a given plane, a field is made of point sources

\[ E(x, y) = \int \int E(x', y') \delta(x - x') \delta(y - y') dx' dy' \]

- Eq 3.40 a-b represent the impulse response of the system (free space or lens)
- Recall linear systems (Chapter 2, page 12, Eq 2.16)
  - Final response (output) is the convolution of the input with the impulse response (or Green’s function)
- Nice! Space or time signals work the same!

\[ \int f(x') \delta(x_0 - x') dx' \Rightarrow f(x_0) \]
3.11 Huygens-Fresnel principle

\[
U(x, y) = \int \int U(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta
\]

\[
U(x, y) = \int \int U(\xi, \eta) e^{\frac{ik}{2} \left[ (x - \xi)^2 + (y - \eta)^2 \right]} d\xi d\eta
\]  \hspace{1cm} (3.41)

Remember:

\[
\begin{array}{c}
\text{Input} \\
\text{Fresnel Diffraction} \\
\text{Output}
\end{array}
\]
3.11 Huygens-Fresnel principle

- Fresnel diffraction equation = convolution
- Fresnel diffraction equation is an approximation of Huygens principle (17th century)

\[
U(x, y) = \frac{1}{i\lambda} \int \int U(\xi, \eta) e^{ikR(\xi, \eta)} \frac{\cos \theta(\xi, \eta)}{R(\xi, \eta)} d\xi d\eta
\]  

(3.42)

! Fresnel is good enough for our purpose

- Note: we don’t care about constants A (no x-y dependence)

\[
U(x, y) = \int \int U(\xi, \eta) e^{ikz(\xi-x)^2+(\eta-y)^2} d\xi d\eta
\]
3.12 Fraunhofer Approximation

- One more approximation (far field)
- The phase factor in Fresnel is:
  \[ \phi(x, y) = \frac{k}{2z} \left[ (x - \xi)^2 + (y - \eta)^2 \right] = \]
  \[ = \frac{k}{2z} \left[ (x^2 + y^2) + (\xi^2 + \eta^2) - 2(x\xi + y\eta) \right] \approx 0 \]

- If \( z \gg k(\xi^2 + \eta^2) \), we obtain the Fraunhofer equation:
  \[ U(x, y) = A \int \int_{-\infty}^{\infty} U(\xi, \eta) e^{-\frac{i2\pi}{\lambda z}(x\xi + y\eta)} d\xi d\eta \]

- Thus eq. 3.44 defines a Fourier transform
- Useful to calculate diffraction patterns!
3.12 Fraunhofer Approximation

- Let’s define:

\[
\begin{align*}
  f_x &= \frac{x}{\lambda z} \\
  f_y &= \frac{y}{\lambda z}
\end{align*}
\]

\[
U(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\xi, \eta) e^{-i2\pi(f_x \xi + f_y \eta)} d\xi d\eta
\]

- Example: diffraction on a slit
3.12 Fraunhofer Approximation

- One dimensional: \( U(x) = \Pi \left( \frac{x}{a} \right) \) = \[
\begin{align*}
a, & \quad |x| < a/2 \\
0, & \quad \text{rest}
\end{align*}
\]

- The far-field is given by Fraunhofer eq:
\[
U(f_x) = \int_{-\infty}^{\infty} U(x) e^{-i2\pi f_x x} \, dx = \frac{1}{2} a \sin \left( \frac{a}{2} f_x \right)
\]

- **Similarity Theorem** + \( \mathcal{F}[\Pi(x)] = \sin c(f_x) \):
\[
U(f_x) = a \sin c(a f_x) = \frac{\sin(a f_x)}{a f_x}
\]

- \( f_x = \frac{x}{\lambda z} \)
3.12 Fraunhofer Approximation

- Always measure intensity $\rightarrow$ the diffraction pattern is:

$$I(f_x) = |U(f_x)|^2 = a^2 \left[ \frac{\sin(af_x)}{af_x} \right]^2$$

(3.48)

- Also Babinet’s Principle

$$f(x) \rightarrow F[\xi]$$

$$1 - f(x) \rightarrow \delta(\xi) - F[\xi]$$
3.12 Fraunhofer Approximation

- **Note:** \( \sin(af_x) = \sin \left( \frac{f_x}{\frac{1}{a}} \right) \rightarrow \frac{1}{a} = \) width of diffraction pattern

- narrow slit: \( a_1 \rightarrow \)

- wide slit: \( a_2 \rightarrow \)

- **Similarity Theorem \leftrightarrow uncertainty principle**
Quiz:

What is the diffraction pattern from 2 slits of size a separated by d?
3.13 Fourier Properties of lenses

- Propagation:

\[
\begin{align*}
U(x_1,y_1) & \xrightarrow{\text{Fresnel}} U(x_2,y_2) \\
U(x_2,y_2) & \xrightarrow{\text{Lens}} U(x_3,y_3) \\
U(x_3,y_3) & \xrightarrow{\text{Fresnel}} U(x_4,y_4)
\end{align*}
\]
3.13 Fourier Properties of lenses

- \( U(x_2, y_2) = A_{12} \iint U(x_1, y_1) e^{\frac{ik}{2d_1} \left[(x_2-x_1)^2 + (y_2-y_1)^2 \right]} \, dx_1 dy_1 \)

- \( U(x_3, y_3) = A_{23} U(x_3, y_3) e^{-i\frac{k}{2f} \left[x_3^2 + y_3^2 \right]} \)

- \( U(x_4, y_4) = A_{34} \iint U(x_3, y_3) e^{\frac{ik}{2d_2} \left[(x_4-x_3)^2 + (y_4-y_3)^2 \right]} \, dx_3 dy_3 \)

Combining Eqs (3.49) is a little messy, but there is a special case when eqs simplify \( \rightarrow \) very useful
3.13 Fourier Properties of lenses

- If $d_1 = d_2 = f$

\[
U(x_4, y_4) = A_4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1(x_1, y_1) e^{-i2\pi(x_1f_x + y_1f_y)} \, dx_1 \, dy_1
\]  

(3.50)

\[
f_x = \frac{x_4}{\lambda f}; f_y = \frac{y_4}{\lambda f}
\]

- Same eq as (3.45); now $z \to f$

- Lenses work as Fourier transformers
  - Useful for spatial filtering

What is $|\mathcal{F}[U]|^2$

\[
\mathcal{F}[|U|^2] = \tilde{\mathcal{F}}[U \cdot U^*] = \tilde{U} \otimes \tilde{U}^*
\]

Autocorrelation
3.13 Fourier Properties of lenses

- **Exercise:** Use Matlab to FFT images (look up “fft2” in help)

- Note the relationship between the frequencies passed and the details / contrast in the final image
Fourier Optics
2D FT pairs $\rightarrow$ diffraction patterns

(Using ImageJ)

FT is in Log(Abs) scale!! Why?
Time-domain: sound

- load your mp3
- plot time-series
- plot frequency amplitude, phase, power spectrum, linear/ log
- show frequency bands, i.e. “equalizer”
- adjust and play in real time

Equalizer from mp3 player
Space-domain: image

- load your image
- display image
- show 2D frequency amplitude, phase, power spectrum, linear/log
- show rings of equal freq., “image equalizer”
- adjust and display in real time- example on next slide
Fourier Filtering (ImageJ)