Appendix A: Introduction to Fourier Transforms

A.1. Linear systems

- Consider the classic problem of a mass on spring
- The spring has a elastic constant
- Body has mass
- Resonant frequency : \( \omega_0 = \sqrt{\frac{k}{m}} \)

The equation of motion is:

\[
\mu \frac{d^2x}{dt^2} + 2\mu \frac{dx}{dt} + \mu \omega_0^2 x = f(t)
\]  \[\text{[A.1]}\]

We can define an "operator" (operates/acts on function)

\[
L = \mu \frac{d^2}{dt^2} + 2\mu \frac{d}{dt} + \mu \omega_0^2
\]  \[\text{[A.2]}\]

\( \Rightarrow \) \( L \) "operates" on \( x(t) \), i.e. the linear forcing position.
$\phi(t)$, we can write Eq. A.1. in very compact form:

$$L \left[ x(t) \right] = f(t) \quad (A.3.)$$

Operator $L$ has important properties:

a) $L(\alpha x) = \alpha \frac{d}{dt}^2 (\alpha x) + \beta \alpha \cdot \frac{d}{dt} (\alpha x) + \mu \cos^2(\alpha x) =$

$$= \alpha \left[ L(x) \right]$$

$$= \alpha \cdot L(x) \quad (A.4.)$$

b) $L(x + \xi) = \mu \frac{d^2}{dt^2} (x + \xi) + \beta \mu \cdot \frac{d}{dt} (x + \xi) + \mu \cos^2(x + \xi) =$

$$= L(x) + L(\xi) \quad (A.5.)$$

**Defining**: An operator $L$ obeying $L(\alpha x) = \alpha \cdot L(x)$

and $L(x + \xi) = L(x) + L(\xi)$ is **linear**.

Many phenomena in nature are linear (or can be approximated as such) = 0 nice analytic solutions

Sometimes we add perturbations to the linear solutions to describe nonlinear phenomena.

4.2.

Notes
4.2. The superposition principle

What happens when 2 forces are applied simultaneously?

First, each force will have a certain "solution" for displacements.

\[
\begin{align*}
\mathbf{m} \frac{d^2 \mathbf{x}_1}{dt^2} &+ \mathbf{F}_1(t) + \mathbf{m} \omega^2 \mathbf{x}_1 = \mathbf{F}_1(t) \\
\mathbf{m} \frac{d^2 \mathbf{x}_2}{dt^2} &+ \mathbf{F}_2(t) + \mathbf{m} \omega^2 \mathbf{x}_2 = \mathbf{F}_2(t)
\end{align*}
\]
\( (A.6) \)

Now, apply both \( \mathbf{F}_1(t) \) and \( \mathbf{F}_2(t) \). What is the displacement \( \mathbf{x} \)?

\[
\mathbf{L} (\mathbf{x}) = \mathbf{F}_1(t) + \mathbf{F}_2(t)
\]
\( \text{Eq. A.6.} \)

\[
\mathbf{L} (\mathbf{x}) = \mathbf{L} (\mathbf{x}_1) + \mathbf{L} (\mathbf{x}_2)
\]

\( \text{linear} \)

\[
\mathbf{L} (\mathbf{x}_1 + \mathbf{x}_2)
\]

\( = 0 \) \( | \mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) | \) \( (A.8) \)

The final solution is the sum of the individual solutions, so this is the superposition principle.

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Notes
A.3. Green's function (impulse response)

Let the acting source be of general form:

\[ f(t) \rightarrow \text{arbitrary shape} \]

\[ f(t) \text{ can be thought of as a succession of short pulses, i.e. Dirac "delta" functions:} \]

\[ f(t) = \sum_{t'} \delta(t-t') \]

\[ \text{property: } \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

Superposition principle: \( f(t) \) is a superposition of \( \delta \)-functions of various delays – therefore the resulting displacement is the superposition of the displacements produced by a \( \delta \)-function, with various delays.

\[ x(t) = \int_{-\infty}^{\infty} f(t') \cdot \delta(t-t') \, dt' \]

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Example A.10. Note that $h(t)$, the response of the system to a short pulse ("impulse"), thoroughly describes the system.  

$h(t)$ is called the "Green's function" or "impulse response" of the system.

→ Complicated problems can be easily tractable using this superposition principle for linear systems.

A.4. The Fourier transform

→ Very efficient way for studying linear systems.

Definition:  
$$\hat{f}[g(t)] = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt$$  

$$= G(f) \quad (A.11)$$

$g(t)$ = the Fourier transform of $f(t)$  
$G(f)$ = the Fourier transform of $g(t)$

$f$ is the "conjugate" variable associated with $t$  
(e.g., $t$ = time, $f$ = frequency).

→ $f : C \rightarrow C$, $f$ must satisfy:

\begin{itemize}
  \item \textbf{a)} $|g(t)| < \infty$ = modulus integrable
  \item \textbf{b)} $g$ has a finite number of discontinuities
  \item \textbf{c)} $g$ has no infinite discontinuities
\end{itemize}

A.S.
The inverse Fourier transform:

\[ \mathcal{F}^{-1} \left[ G(f) \right] = \int_{-\infty}^{\infty} G(f) \cdot e^{i\omega f} \, df \]

\[ = g(t) \]  \hspace{1cm} \text{(A.12.)}

i.e. \[ \mathcal{F}^{-1} \left[ \mathcal{F} (G) \right] = g \]  \hspace{1cm} \text{(A.13.)}

Meaning of the F.T: A given signal \( g(t) \) can be reconstructed as a summation of sinusoids \( e^{i\omega f} \) weighted properly \( \left[ b \cdot G(f) \right] \).

? Is the F operator linear:

\[ \mathcal{F} \left[ a \cdot g_1(t) + b \cdot g_2(t) \right] = \]

\[ = \int_{-\infty}^{\infty} \left[ a \cdot g_1(t) + b \cdot g_2(t) \right] \cdot e^{-i\omega f} \, df = \]

\[ = a \int_{-\infty}^{\infty} g_1(t) \cdot e^{-i\omega f} \, dt + b \int_{-\infty}^{\infty} g_2(t) \cdot e^{-i\omega f} \, dt = \]

\[ = a \mathcal{F} \left[ g_1(t) \right] + b \mathcal{F} \left[ g_2(t) \right] \]  \hspace{1cm} \text{(A.14.)}

\[ \Rightarrow \text{Yes, } \mathcal{F} \text{ is a linear operator} \]

A.6.

Notes
A.S. Fourier transform properties

a) Shift theorem

Let \( G(f) = \mathcal{F}[g(t)] \)

\[ \text{What is the \( \mathcal{F}.T. \) of } g(t - t_0) \]

\[ \mathcal{F}[g(t - t_0)] = G(f) \cdot e^{-j2\pi f t_0} \quad (A.15) \]

So, shift in one domain corresponds to linear phase (ramp) in the other, i.e. Fourier domain.

b) Parseval's theorem

\[ \text{Let } G(f) = \mathcal{F}[g(t)] \]

\[ \int_{-\infty}^{\infty} (|g(t)|^2) dt = \int_{-\infty}^{\infty} |G(f)|^2 df \quad (A.16) \]

- Conservation of energy.
c) Central Limit Theorem

If \( g \xrightarrow{\mathcal{F}} G \)

\[ g(\omega) = \left\{ \begin{array}{l l}
\infty & \text{if } \omega = 0 \\
G(\omega) & \text{otherwise}
\end{array} \right. \quad (A.17) \]

Useful for normalizing actual experimental data.

d) Similarity Theorem

Let \( G' \) be the F.T. of \( g(\omega) \), i.e.

\[ g \xrightarrow{\mathcal{F}} G \]

\[ F \left[ g(\omega \tau) \right] = \frac{1}{|\tau|} \cdot G \left( \frac{\omega}{\tau} \right) \quad (A.18) \]

- Eq. A.18. provides intuitive feeling for the F.T.'s,

  - A broader function in one domain (e.g., time)
  - gives narrower function in the other domain (e.g., freq).

Notes

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A.8.
Example: To obtain short temporal pulses of light, one needs a broad spectrum (Fresnel zones).

Only an infinitely broad spectrum allows for $\delta$-function pulses.

$\mathcal{F}(f)$

$\mathcal{F}^{-1}$

$\delta$-function

$\rightarrow$ physically impossible
• The next FT properties refer to the operations of "convolution" and "correlation" (2-page discussion)

- Let's define them:
  \[ g(t) \xrightarrow{F} G(f) \]
  \[ h(t) \xrightarrow{F} H(f) \]

• Convolution of \( g \) and \( h \) (\( \odot \)):
  \[ g \odot h = \int_{-\infty}^{\infty} g(t') \cdot h(t'-t) \, dt' \]  \( (A, 11) \)

• Correlation of \( g \) and \( h \) (\( \otimes \)):
  \[ g \otimes h = \int_{-\infty}^{\infty} g(t') \cdot h'(t'-t) \, dt' \]  \( (A, 20) \)

The difference between \( \odot \) and \( \otimes \) is \( h(x-x') \) vs. \( h(x'-x) \)
i.e. flip vs. non-flip of function \( h \).
→ If \( h(t-t') = h(t'-t) \), i.e. \( h \) is even function
  \( \Rightarrow \) \( \odot \) and \( \otimes \) are the same.

\( g \otimes g \equiv \text{autocorrelation} \)

**Exercise**: Show that:
\[ \square \otimes \square = \square \]
\[ \prod \otimes \prod = \prod \]
\[ \text{Gaussian} \otimes \text{Gaussian} = \text{Gaussian} \]

A. 10.
Let recall the proposition principle (Green’s function), i.e., Eq. A.10.

\[ x(t) = \int_{-\infty}^{t} \delta(t') \cdot h(t-t') \, dt' \]

Using the definition in Eq. A.19:

\[
\Box \begin{align*}
X(t) &= \delta(t) \ast h(t) \\
\end{align*}
\]

(A.21.)

where

\- \delta = \text{impulse produced by } F

\- F = \text{arbitrary force}

\- h = \text{impulse response}

Note:

\- x = \text{output}

\- F = \text{input}

\- h = \text{property of the system—linear system}

```
\[
F \rightarrow h \rightarrow X
\]
```

\- Eq. A.21 states that:

"The output of a linear system to an arbitrary input is the convolution of that input with the system's impulse response."

A.11.
e) **Convolution Theorem**

\[ \mathcal{F} \left[ g \ast l(t) \right] = \mathcal{F}[g] \cdot \mathcal{F}[l] \quad \text{(#A.22)} \]

i.e., \( \mathcal{F} \left[ \int_{-\infty}^{\infty} g(t') \cdot l(t-t') \, dt' \right] = G(f) \cdot L(f) \)

Convolution in one domain corresponds to product in the other domain. **GREAT!**

Multiplication is easy to do.

→ Let’s apply this to solve the Green problem (#A.21)

\[ x(t) = f(t) \ast l(t) \]

Take \( f, l = 0 \) \( \to \quad x(0) = \mathcal{F} \left[ f(t) \ast l(t) \right] \)

Apply Eq. (A.2) \( \Rightarrow \quad X(0) = F(0) \cdot L(0) \quad \text{(A.23)} \)

Most linear problems should be solved in the Fourier domain.

→ Here’s how it works.

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**Notes**
Typical problem:

- We know the system, characterized by \( h(t) \).
- The input, \( f(t) \).
- We want the output, \( x(t) \).

Solution:

\[
\begin{align*}
   x(t) & \underset{\mathcal{F}}{\rightarrow} X(f) \\
   f(t) & \underset{\mathcal{F}}{\rightarrow} F(f)
\end{align*}
\]  

(Eq. A.24,)

Eq. A.23 is the solution in the frequency domain:

\[
x(f) = F(f) \cdot H(f) \tag{A.25}
\]

Final solution, i.e., \( x(t) \) in time domain:

\[
   x(t) = \mathcal{F}^{-1} \left[ X(f) \right] \tag{A.26}
\]

\( h(t) \equiv \text{impulse response} \)

\( H(f) \equiv \text{transfer function} \).

Complicated problems can be solved using the procedure of (A.24 - A.26), i.e., \( \mathcal{F} \)'s and multiplications - PC's do these questions very fast.

A.13.
f) **Correlation theorem**

\[
F \left[ g \otimes h \right] = G \cdot H^* \quad (A.27.)
\]

\[
F \left[ \int_{-\infty}^{\infty} g(t) \cdot h(t-t') \, dt' \right] = G(f) \cdot H(f)^* \quad \text{"H-conjugate"}
\]

*Particular case: \( g = h \) = \( \Rightarrow \) auto-correlation.

\[
F \left[ g \otimes g \right] = G \cdot G^* = |G|^2 \quad (A.28)
\]

*F.T. of an auto-correlation is the power spectrum.*

(Important for both time- and space-variant signals.)

g) **Differentiation theorem**

\[
\frac{\partial}{\partial t} \left[ G(f) \right] = \int_{-\infty}^{\infty} \left[ -i2\pi ft \right] G(f) \, df \quad (A.29)
\]

Take the time derivative:

\[
\frac{\partial g(t)}{\partial t} = \int_{-\infty}^{\infty} \left[ -i2\pi ft \right] G(f) \, df
\]

\[
= \int_{-\infty}^{\infty} \left[ + i2\pi ft \right] G(f) \, df
\]

\[
= \frac{d}{dt} \left[ F \left[ g(t) \right] \right] = 12\pi f \cdot G(f) \quad (A.30)
\]

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**Notes**
\[
\frac{d^2 g(t)}{dt^2} = \frac{d}{dt} \left[ \frac{d g(t)}{dt} \right]
\]

\[
= \frac{d}{dt} \left[ i \cdot \frac{d g(t)}{dt} \right] = i \cdot \frac{d^2 g(t)}{dt^2} \cdot G(j)
\]

\[
= - (2\pi f)^2 \cdot G(j)
\]  \hspace{1cm} (A.31)

**General Solution:**

\[
\frac{d^n x(t)}{dt^n} = i^n (2\pi f)^n \cdot X(j)
\]  \hspace{1cm} (A.32)

Eq. A.32 expresses the **differentiation theorem**.

**Great tool for solving linear differential equations.**

**Example:**

\[
a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = d(t)
\]

Take \( F \), \( T \):

\[
- a (2\pi f)^2 x(f) + b \cdot i \cdot 2\pi f \cdot x(f) + c x(f) = d(f)
\]

Solve for \( x(f) \), i.e., solution in the frequency domain.

\[
x(f) = \frac{d(f)}{-a(2\pi f)^2 + i \cdot 2\pi b \cdot f + c}
\]  \hspace{1cm} (A.33)

\[
\Rightarrow \text{Time-domain solution: } x(t) = \mathcal{F}^{-1} \{ x(f) \}
\]  \hspace{1cm} (A.34)

**Notes**

\[\text{Very useful}\]